ensure actual realization. There arise at once questions of biological adaptation, of vital tenacity and of purposeful action. Appeal to the record of the animal races reveals in some cases a marvelous endurance, in others the briefest of records, while the majority fall between the extremes. Many families persisted for millions of years. A long career for man may not therefore be denied on historical grounds, neither can it be assured; it is an individual race problem; it is a special case of the problem of the races in the largest sense of the phrase.

But into the problem of human endurance two new factors have entered, the power of definite moral purpose and the resources of research. No previous race has shown clear evidence that it was guided by moral purpose in seeking distant ends. In man such moral purpose has risen to distinctness. As it grows, beyond question it will count in the perpetuity of the race. No doubt it will come to weigh more and more as the resources of destructive pleasure, on the one hand, and of altruistic rectitude on the other are increased by human ingenuity. It will become more critical as the growing multiplicity of the race brings upon it, in increasing stress, the distinctive humanistic phases of the struggle for existence now dimly foreshadowed. It will, beyond question, be more fully realized as the survival of the fittest shall render its verdict on what is good and what is evil in this realm of the moral world.

But to be most efficient, moral purpose needs to be conjoined with the highest intelligence, and herein lies the function of research. None of the earlier races made systematic inquiry into the conditions of life and sought thereby to extend their careers. What can research do for the extension of the career of man? We are witnesses of what it is beginning to do in rendering the forces of nature subservient to man's control and in giving him command over the maladies of which he has long been the victim. Can it master the secrets of vital endurance, the mysteries of heredity and all the fundamental physiological processes that condition the longevity of the race? The answer must be left to the future, but I take no risk in affirming that when ethics and research join hands in a broad and earnest endeavor to compass the highest development and the greatest longevity of the race the era of humanity will really have begun.

T. C. CHAMBERLIN

THE THESIS OF MODERN LOGISTIC[1]

I HAVE chosen to report upon this subject because it is one in which I have found no little interest in recent years; because the thesis in question represents one among the greatest of all the triumphs of critical thought; because it possesses such high and permanent importance as belongs to intellectual activity above the levels of workaday life; because it is sufficiently new, timely and general in its appeal; and finally because, whilst it has come to be everywhere a topic of much philosophic and scientific allusion, but relatively few, it seems, have been at the pains to ascertain what the thesis precisely is.

To tell what it is, to render it intelligible not merely to astronomers and mathematicians but also to that larger class of educated folk who, as their primary interests lie elsewhere, are not accustomed to thinking much about the fundamental subtleties of logic and mathematics—*that* is one of the two aims of this address; the other one being to present, in so far as time will

[1] Address of the vice-president and chairman of Section A—Mathematics and Astronomy—American Association for the Advancement of Science, Boston, 1909.

allow, the more salient among the facts by which the thesis is supported.

It is no part of my purpose to treat the matter historically. As, however, the thesis in question is the goal and culmination of two originally independent but closely related and finally convergent movements of modern thought, I can not refrain from saying a brief preliminary word regarding each of them. They may be characteristically designated as the critico-mathematical movement and the logistical movement.

The distinctively critical spirit is not a new manifestation in mathematics. The age of Euclid was a critical age. And just now, thanks to the superb edition of the "Elements" by Dr. Heath with its wonderful richness of bibliographic citation, quotation and critical commentary, one is enabled to understand better than ever before how very fine and penetrating in fundamental questions of geometry and of logic was the thought of the age that produced the Alexandrine classic—the age, I say, for the "Elements" is to be attributed not less to the age of Euclid than to Euclid the man. But it is not of antiquity that I wish to speak. I refer to the critical movement in *modern* mathematics—to the demand for precision of concept, to the process of logical rigorization, to the sense and the craving for perfection of intellectual and scientific form, in a word, to that spirit of creative criticism which, following close upon the great Eulerian and pre-Eulerian period of discovery, manifesting itself already in the works of Gauss and Lagrange, finding powerful agencies in the analytic genius of Cauchy and Bolzano, in the geometric genius of Lobachevski and Bolyai, waxed in intensity throughout the lapsing decades of the nineteenth century, at length pervading the entire realm of mathematics like a refining and purifying fire. The result of this critical movement, thus originating in mathematics and conducted by mathematicians, was, not indeed the grounding of mathematics itself, regarded as a unitary science, but the grounding rather, upon distinct bases of postulated mathematical notions and propositions, of various great *branches* of the science; in witness whereof—to cite but one example—behold the theory of the real variable as founded by Weierstrass upon the familiar theory of the cardinal numbers assumed as certain, primordial and fundamental.

Such bases, however, were destined to appear, in the light of modern researches in another field or in what seemed at all events another, namely, the field of logic, not as constituting the foundation either of mathematics or of any of its branches but as genuine components of the superstructure. For it has ever been the faith of the logician that there are a few ideas in terms of which all definable ideas admit of immediate or mediate definition and a few propositions upon which as a basis or from which as a body of premises all demonstrable propositions admit of proof or deduction; and it has ever been the chief of the logician's problems to discover such a system of primitive concepts and propositions. It is in nothing less than a closely approximate solution of that hoary problem that modern investigations in logic have culminated. As every one knows, the conception of logic as an autonomous science is nothing new. Among the very greatest contributions of antiquity to human knowledge is the "Logic" of Aristotle. As a scientific achievement it is comparable to the "Elements" of Euclid—comparable to it also in another respect, namely, that it was not significantly improved upon for nearly two thousand years. Though always indispensable as an instrument of thought, yet logic, regarded as a science, remained stationary for so long a time, showing no

token of life, that it came to be thought of as a thing that is dead. And I suspect that even to-day there may be found scientific men of eminence who are not aware of the fact that in our time logic, as a field of research, affords a spectacle of teeming activity quite as intense as may be witnessed in physics, for example, or in astronomy or biology—men, it may be, who have yet to learn that, owing to modern logistic research, it would be as radical an error to identify the modern significance of the term logic with that of the Aristotelian system as to identify the modern meaning of the term geometry with that of Euclid's "Elements" or to identify modern jurisprudence with the code of Lycurgus or the "Pandects" of Justinian. By the logistical movement I mean the movement that began—somewhat prematurely, however, as the event was destined to show—in the logical speculations and investigations of Jungius (1587–1657), Leibniz (1646–1716) and Lambert (1728–1777); awaited the powerful impulse imparted by Boole's symbolical "Investigation of the Laws of Thought" (1854); and, under the leadership of C. S. Peirce in our own country, of Schröder in Germany, of Peano and his numerous collaborators in Italy, of Couturat, brilliant expounder and advocate of the subject in France, and of Russell, Whitehead and McColl in England, has at length produced that imposing body of doctrine now known throughout the scientific portions of the world under the characteristic name of symbolic logic.

In its present form and state of development this science is constituted of three distinct but interconnected branches: the logic of classes, which, though it corresponds to the traditional system of Aristotle, is far from being identical with it; the logic of propositions; and the logic of relations, which was originated by Charles S. Peirce, was much elaborated, refined and clarified by Schröder in the third volume of his "Vorlesungen über die Algebra der Logik," 1895, but owes its present form and conception mainly to the various contributions of Bertrand Russell in recent volumes of the *Revue des Mathématiques* (formerly the *Revista di Matematica*) and elsewhere.

For the purpose in hand the thing to be noted is the discovery of the fact that for the *notional basis* of the triple organon it was necessary and *sufficient* to assume, without definition, a very few notions—called the primitive ideas, or constants, of logic—in order that in terms of them all other notions entering logic should be definable; and that it was necessary and sufficient, for the *propositional basis,* to assume, without proof, a somewhat larger yet very small number of propositions—called the primitive propositions, or the premises, of logic—in order that by means of them all other propositions of the science should be capable of demonstration. This is not all, however; for it has been found—and here we encounter the *thesis of modern logistic,* the common culmination and result of the two movements hitherto sketched, and so a joint achievement of the logician and the mathematician, though hardly foreseen by either of them—it has been found, I say, that the basis of logic is the basis of mathematics also—that, in other words, given the primitives of logic, mathematics requires none of its own but that in terms of the logical primitives all mathematical ideas and all mathematical propositions admit respectively of precise definition and of rigorous demonstration. Accordingly, if a scientific edifice may properly be regarded as consisting of both foundation and superstructure, it becomes evident, the thesis once established, that, instead of logic and mathematics being, as hitherto supposed, radically distinct sciences, the latter is strictly

the outgrowth and prolongation of the former, and that the twain are one as the branches and upper stem of a tree are continuous with the lower stem and the roots.

To any one who knows something of the immensity of modern mathematics, something of the continent of doctrine that the term connotes, something of the countless variety and the infinite complexity of the ideas and propositions that compose the body and constitution of the science, the simple thesis in question is really astounding. And one demands that the thesis be explicated in terms in order that one may know precisely and concretely in detail what it constates. What, we wish to be informed, *are* the logical primitives that, it is alleged, are capable, though so few, of supporting so great a burden? Before attempting to meet this demand, I beg to remind you of the fact that, given a logically coherent or autonomous body of propositions, it is always in some degree a matter of arbitrary choice, though probably never one of complete indifference which of the propositions are taken as fundamental and which as derivative—that is, which are assumed and which proved. In every case the choice is to be guided by considerations of expedience, of interest, or of economy, but seems never to be coerced by necessity or by "the nature of things." Questions of relative interest, however, and of relative expedience and economy are matters of judgment. Accordingly it is not a matter for surprise that several systems of logical primitives have been devised and submitted, differing any two of them in respect of one or more elements but agreeing all of them as to the adequacy of a small number of elements, and that among investigators in the field it remains a moot question which of the systems, if any one of them enjoys that distinction in comparison with the rest, is to be preferred.

The system that I shall present here is that which Russell has adopted in his great synthesis of modern logic and modern mathematics, "The Principles of Mathematics," and which with slight modifications has been so delightfully expounded by Couturat in his "Les Principes des Mathématiques" and his "Traité de Logistique." I have thought it best to gather together all the primitive elements of the three branches of logic for compact presentation in a single uninterrupted list under their appropriate headings, reserving commentary for a subsequent stage. Moreover, despite the somewhat forbidding appearance, at first glance, of logical symbolism, I have decided to present primitive propositions in symbolic form, employing for this purpose the symbolism of Peano slightly modified by selection from that of Schröder. Indeed this symbolism is not difficult to master; and if at first it seems a thing of so frightful mean that to be hated needs but to be seen, yet, seen often enough to become familiar with its face, we come first to endure, and then to embrace it as a convenient and potent means of clarity and economy alike of thought and of expression. It is a moot question which one, if indeed any one, of the three varieties of the logical calculus is primordial to the other two. As, however, discourse of any kind, whether about classes or about relations, would seem to be difficult if not impossible without propositions, I shall follow the leading of common sense and begin with

The Logic of Propositions.—In addition to the notions, truth and its negative, which, though they are constantly employed, seem neither to admit of effective definition nor to be strictly coordinate with any other indispensable notion, the primitive notions in propositional logic are

(1) Material Implication,
(2) Formal Implication.

And the primitive propositions are

(1) $p\supset q.\supset.p\supset q$,
(2) $p\supset q.\supset.p\supset p$,
(3) $p\supset q.\supset q\supset q$,
(4) If $p\supset q$ and if p be true, p may be dropped and q asserted,
(5) $p\supset p.q\supset q.\supset.pq\supset p$,
(6) $p\supset q.q\supset r.\supset.p\supset r$,
(7) $q\supset q.r\supset r.p\supset.q\supset r:\supset.pq\supset r$,
(8) $p\supset p.q\supset r.pq\supset r:\supset:p\supset.q\supset r$,
(9) $p\supset q.p\supset r.\supset.p\supset qr$,
(10) $p\supset p.q\supset q\ \supset:(p\supset q)\supset p.\supset p$,

in which, as elsewhere, p, q and r denote propositions, $\supset$ (inverse of the letter c) stands for the word *implies*, pq means "p and q," while the points or dots serve the double use of denoting the word *and*, like the first dot in (5), or, like those in (1), playing the rôle of parentheses in indicating the relative ranks of the various parts of a formula. Thus, for example, (7) may be translated to read, the proposition "q implies q and r implies r and p implies that q implies r" implies the proposition "p and q together imply r"; or, in hypothetic form, if q implies q, and r implies r, and p implies that q implies r, then p and q together imply r.

The Logic of Classes.—The primitive notions in this calculus are

(1) Proportional Function, denoted by such symbols as $\phi(x)$, $\Psi(x)$, etc.,
(2) The Relation (denoted by ϵ, read *is* or *belongs to*) of an individual to a class (containing it),
(3) The notion *such that*, denoted by $\ni$ (inverse of the Greek letter ϵ).

And the primitive propositions are

(1) $k\epsilon\}x\ni\phi(x)\}\supset\phi(k)$,
(2) $\phi(x)=\Psi(x).\supset:x\ni\phi(x).=.x\ni\Psi(x)$.

The Logic of Relations.—In this calculus, which Russell has shown to be the logic *par excellence* of mathematics, the primitive notions are

(1) Relation, denoted as a class by rel and as individuals by such capitals as R, R', etc.,
(2) Identity, denoted by the symbol 1'.

The primitive propositions are

(1) $R\epsilon\mathrm{rel}.\supset:xRy.=.x$ has the relation R to y,
(2) $R\ni\mathrm{rel}\ \supset\exists\,\mathrm{rel}\frown R'\ni(xR'y.=.yRx)$,
(3) $\exists\,\mathrm{rel}\frown R\ni(\rho=\iota x.\rho=\iota y)$,
(4) $\smile{}'K\epsilon\mathrm{rel}$,
(5) $\frown{}'K\epsilon\mathrm{rel}$,
(6) $R_1R_2\epsilon\mathrm{rel}$,
(7) $-R\epsilon\mathrm{rel}$,
(8) $\epsilon\epsilon\mathrm{rel}$,
(9) $1'\epsilon\mathrm{rel}$,
(10) $x1'x$,
(11) $1'\supset\breve{1}'$,
(12) $R\epsilon\mathrm{rel}.xRy.y1'z\ \supset.xRz$.

To the foregoing primitives must be added the notion of *denoting*, which has been made the topic of a most subtle and luminous discussion by Russell in the fifth chapter of the work above cited. The notion is that of the sense in which an individual is denoted by a concept that occurs in a proposition that is not a proposition about the concept, as "She bought *a beautiful gown*"—the thing purchased be nothing so tenuous and translucent as the concept, a beautiful gown, but presumably a concrete thing reasonably opaque.

By way of elucidating the foregoing and further sketching out the three divisions of logic, I shall now proceed to give some explanation of the primitive terms and a statement of the principal definitions and theorems composing them.

Definitions and Theorems in Propositional Logic.—The central term, proposition, is defined in terms of (material) implication, namely, a proposition is that which implies itself. The two varieties of implication are often confused and the distinction between them, being difficult to draw sharply and clearly, is to be acquired very much as a child learns to distinguish

cats from dogs. For one thing material implication subsists only between propositions while formal implication, though it is present in propositional logic, holds only between propositional functions. Now a proposition to be such must be true or else false, while a propositional function, say, x is a number, though it has the form of a proposition is not one, being neither true nor false, until the unspecified term or terms (x in the example cited) are specified and then we have no longer a function but a proposition. The implication postulated in the primitive propositions is material. The meaning of (1) is that if $p \supset q$, then $p \supset q$ is a proposition; (2) means that whatever implies anything is a proposition; and that of (3) is, whatever is implied is a proposition. Number (4), which does not admit of completely symbolic statement, is the postulate that justifies the advance from the hypothetic to the categoric—the advancement involved in passing from saying "such and such a conclusion is true *if* the premises are true" to saying, once the premises are granted true, "the proposition" (not now regarded as a conclusion) "*is* true."

One of the most striking facts in the propositional logic is the theorem that every false proposition implies all propositions and that all true propositions are implied by every proposition. The shocking character of the theorem—which refers, of course, to material implication only—disappears on reflecting that the proposition, p implies q, means simply "q or not-p"—means, that is, "q is true or p is false" and *nothing else;* for surely it is nothing shocking to affirm that a proposition that is not contradicted by any proposition in the class of true propositions is a member of the class; and that affirmation seems equivalent to asserting that "p implies q" is true unless q is false and p true. If you assert of two propositions p and q that p implies q, thereby meaning simply and solely that q can not be false and p true, then unless it happens that at once q is false and p true, there would seem to be in the arsenal of refutation no weapon with which your assertion may be struck down. The primitive propositions are some of them far from being "self-evident." It is not essential that they should be. They are chosen with reference to their sufficiency and look for justification to the body of their consequences. In these they shine—not *a priori* but *a posteriori.* Neither can they be proved true by deducing them from a theorem that is itself deduced from them—to say which is, of course, but to utter a commonplace. As an exercise, however, it is legitimate as well as interesting and instructive to assume the foregoing theorem as a postulate and as such to apply it as a test to the primitive propositions in question. Thus, to take a single example, the procedure in the case of (8) would be as follows. Let r be true and p and q either or both be false or true; then $q \supset r$ is true, hence $p \supset . q \supset r$ is true, hence (8) is true. Let r be false and p and q be true; then $p \supset p$ and $q \supset r$ are both true, pq is true, $pq \supset r$ is false, hence what precedes the colon is false, hence (8) is true. And so on for the remaining possible suppositions respecting p, q and r.

Two propositions are *equivalent* if each implies the other, and we write $p = q$. Two propositions are equivalent when and only when both are true or both are false. The fundamental operations of propositional *multiplication* and *summation* are definable as follows: We may first define the *logical product* of the two *special* propositions—a is a proposition, b is a proposition—to be the proposition, a is a proposition and b is a proposition. Then, denoting this special product by $a \supset a . b \supset b$,

the logical product, pq or $p\frown q$, of *any* two propositions, p and q, may be formally defined by the definition:

$$p\supset p.q\supset q.r\supset r.\supset:.pq =\ :p\supset(q\supset r).\supset r.$$

This definition of the notion—vulgarly called the joint assertion of p and q—may be rendered thus: p, q, r being propositions, the product of p and q is the proposition—any proposition r such that p implies that q implies it, is true. The *logical sum*, $p\smile q$, of two propositions p and q admits of the definition:

$$p\supset p.q\supset q.r\supset r.\supset:p\frown q =\ :p\supset r.q\supset r.\supset r\ ;$$

that is, p, q and r being propositions, $q\smile p$ is the proposition equivalent to the proposition that r is implied by the product of $p\supset r$ and $q\supset r$. Such is the definition of the phrase, p or q. It is noteworthy that, whilst pq is true when and only when p and q are *both* true, the sum $p\smile q$ is true whenever *either* p or q is true. Among cardinal theorems I will, further, mention the laws of tautology, commutation, association and distribution:

$$pp(\text{or } p^2) = p, \qquad p\smile p = p\ ;$$

$$p\frown q = q\frown p, \qquad p\smile q = q\smile p\ ;$$

$$(p\frown q)\frown r = p\frown(q\frown r), \qquad (p\smile q)\smile r = p\smile(q\smile r);$$

$$p\frown(q\smile r) = (p\frown q)\smile(p\frown r),$$

$$p\smile(q\smile r) = (p\smile q)\smile(p\smile r) = p\smile q\smile p\smile r.$$

The *negative*, $-p$, of p is a proposition definable thus:

$$p\supset p.q\supset q.\supset: -p = .p\supset q,$$

which states that $-p$ is the proposition equivalent to the proposition that p implies all propositions; and we have the theorem of *double negatives*: $-(-p) = p$. Also the theorems of contradiction and excluded middle: $-p\frown q$ is false; $-p\smile q$ is true.

Definitions and Theorems in Class Logic.—As already pointed out, a propositional function—say, x is a pragmatist, or $\tan x = y$—though a proposition in form, is not one in fact, being neither true nor false. But such a function yields a proposition whenever the indeterminate terms, as x, y, are replaced by determinate terms. Thus any such function is a sort of envelope of a limitless number of propositions. A function being given, those terms that on being substituted for its indeterminates yield true propositions are said to constitute a *class*. The symbolism $x\ni\phi(x)$ means "the class of terms x such that $\phi(x)$ is true," and primitive proposition (1) asserts that, if the individual k is a member of the class, $\phi(k)$ is true. Two functions $\phi(x)$ and $\Psi(x)$ are said to be *equivalent* when the propositions of every pair of propositions obtainable by substituting definite terms for x are equivalent; and (2) states that when two functions are equivalent the corresponding classes are the same—composed of the same individuals. If the propositions derivable from $\phi(x)$ are all of them false, the function is said to determine a *null-class;* and it readily follows that all null-classes are *extensionally* the same, so that we can, in this sense, speak of *the* null-class. The definition and symbolic expression of "x is identical with y," x and y being individuals, is $x = y. =: x\epsilon u.\supset_u.y\epsilon u$, where $\supset_u$ means "implies for every (class) u." The relation in question is *symmetric*, a fact involved in the theorem, $x = y. = .y = x$. A *singular* class u (class of but one term) is defined to be such that

$$x\epsilon u.y\epsilon u.\supset.x = y\ ;$$

and a singular class u is symbolically distinguished from its term a by writing ιa to denote u, and ιu to denote a; so we have $\iota a = u$, $\iota u = a$, and $\iota\iota u = u$, but not $u = a$. The notion of *inclusion* of the terms of a class u by a class v is denoted by $u\supset v$ (where $\supset$ is the symbol for "implies" in propositional logic) and is defined to be

such that $u \supset v . = : x \epsilon u . \supset . x \epsilon v$. Two classes u and v are (extensionally) *identical,* and we write $u = v$, when and only when $u \supset v$ and $v \supset u$. Two classes are *disjoint* if neither includes a term of the other. It is necessary to avoid confounding ϵ with the use of $\supset$ in class logic, the former holds between an *individual* and a class but $\supset$ holds only between classes. Thus, if class $u \supset$ class v, and if individual $a \epsilon u$, we can not write $a \supset v$.

The important notions of class *multiplication* and *summation* are definable as follows. The logical *product* of the classes u and v, which is denoted by $u \frown v$, is such that $u \frown v = . x \ni (x \epsilon u . x \epsilon v)$; while the logical *sum,* $u \smile v$, u and v being disjoint or not, is such that $u \smile v . = . x \ni (x \epsilon u . \smile . x \epsilon v)$. Among cardinal theorems are the laws of *tautology, commutation, association, distribution* and *double negation:*

$$u \frown u = u = u \smile u ;$$

$$u \frown v = v \frown u, \quad u \smile v = v \smile u ;$$

$$u \frown (v \frown w) = (u \frown v) \frown w, \quad u \smile (v \smile w) = (u \smile v) \smile w ;$$

$$u \frown (v \smile w) = (u \frown v) \smile (u \frown w),$$

$$u \smile (v \frown w) = (u \smile v) \frown (u \smile w);$$

and $-(-u) = u$, where $-u$, called the negative of u, is, by definition, such that $-u . = . x \ni (x - \epsilon u)$.

The foregoing sketch indicates how the class logic sends its roots down into the soil of the propositional logic, and there is at the same time exhibited a remarkable parallelism between the two logics. It is important, however, to note the fact, pointed out by Schröder, that the parallelism is not thoroughgoing. For example, if p, q, r be propositions and a, b, c be classes, we have

$$pq \supset r . = : p \supset r . \smile . q \supset r,$$

but *not*

$$a \frown b \supset c . = : a \supset c . \smile . b \supset c .$$

Explanations, Definitions and Theorems in Relational Logic.—In its present form this calculus is mainly the creation of Mr. Bertrand Russell. It was he who perceived and demonstrated the advantage of adopting the extensional as distinguished from the intensional view of relations. It was he who perceived and demonstrated its preeminent importance in and for mathematics. Finally, it was he who cast its general principles—primitive propositions, fundamental definitions, theorems and their proofs—in symbolic form (cf. *Revue de Mathematiques*, vol. 7, 1900–1901).

In order to understand the doctrine including its primitive propositions above given, it will be necessary to explain or define the principal concepts involved in it and to associate with them the symbols (including those already explained) by which they are denoted. These concepts and symbols are as follows, the numbers (1), (2), $\cdots$ referring to primitive propositions. The writing xRy means to assert that x has the relation R to y, so that a relation has *sense* or direction; the symbols ρ and $\breve{\rho}$, called respectively the *domain* and the *codomain* of R, denote respectively the classes of terms that may stand before R and after R; the logical sum of these classes is the *field* of R; if x be a term of ρ, $\breve{\rho}x$ denotes the class of terms y such that xRy, and if x be a term of $\breve{\rho}$, ρx is the class of terms y such that yRx; a class is said to *exist* unless it be a null-class, and the *existence* of a class is affirmed by writing $\exists$ before its symbol, as in (3); if u is a class of terms of ρ, $\breve{\rho}u$ is the class of terms y such that, given any one of them, there is in u an x for which xRy; on the other hand, u again being a class of terms of ρ, $u\breve{\rho}$ denotes the class of terms y such that for *every* term x of u we have xRy; if, now, u is a class of terms in the codomain $\breve{\rho}$, ρu denotes the class of terms such that, given any one y of them, there is in u a term x for which yRx, while, on

the other hand, $u\rho$ is the class of terms such that, given any one y of them, we have, for *every* x of u, yRx; R is said to be *included in* R', $R \supset R'$, if and only if, for all x's and y's, xRy implies $xR'y$; and R and R' are *equivalent* when and only when each of them includes the other; (2) asserts that, given any R, there is a relation R'—called the converse of R and denoted by $\breve{R}$—such that xRy and $yR'x$ are equivalent functions; a relation R is said to be *symmetric* when and only when $R = \breve{R}$; (3) affirms that, given any two terms x and y, there is between them a relation that does not subsist between the terms of any other pair of terms; the logical *sum*, $R_1 \smile R_2$, of two relations R_1 and R_2 is a relation such that the proposition $x(R_1 \smile R_2)$ is equivalent for all x's and y's to the logical sum of the propositions xR_1y, xR_2y; the logical product, $R_1 \frown R_2$, is such that $x(R_1 \frown R_2)y$ is equivalent to the product $xR_1y.xR_2y$, for all x's and y's; if K be a class of relations, their *sum*, $\smile$‘K, affirmed by (4) to be a relation, is a class of relations such that, given any one R of them and any pair x, y for which xRy, there is in K a relation R' for which $xR'y$, and that, given any R' of K and a pair x, y for which $xR'y$, there is in the sum-class an R for which xRy; similarly the *product*, $\frown$‘K assumed by (5) to be a relation, is the class of relations such that, R being any one of them and x and y being a pair for which xRy, then, for *every* R' of K, $xR'y$, and conversely, if x and y be a pair for which $xR'y$ holds for every R' of K, there is in the product-class an R for which xRy; R_1 and R_2 being relations, their *relative product*, R_1R_2, affirmed by (6) to be a relation, is defined to be such that, if xR_1R_2z, there is a y for which xR_1y and yR_2z, and that, if xR_1y and yR_2z, then xR_1R_2z; R^2 means RR; a relation R is transitive if and only if R^2 is included in R, that is, if the product of xRy and yRz implies xRz; R being a relation, its *negative*, $-R$, affirmed by (7) to be a relation, is defined to be such that, $x - Ry$ is true or false according as xRy is false or true; if y is a class of classes, their *sum* ‘y is the class of terms x such that $x\epsilon^2y$; *diversity*, 0’, is defined to be the negative of identity, so that $0' = -1'$; R is a *uniform* relation, $Nc \rightarrow 1$, when and only when, whatever x of ρ be given, there is one and but one y for which xRy; R is a *couniform* relation, $1 \rightarrow Nc$, when $\breve{R}$ is uniform; R is a *biuniform* relation, $1 \rightarrow 1$, when it is both uniform and couniform.

Such are the chief of the concepts in the superstructure of the logic of relations. In the study of relations one is close to reality. We do not say with Hegel "Das Seyn ist das Nichts" but rather with Lotze "Being consists in relations." The realm of the thinkable is filled by a multidimensional tissue of relations. These are finer than gossamer but stronger than cables of steel. Among the *theorems* of the general theory the following, which are readily proved by means of the symbolic machinery, are cardinal. Each relation R has one and but one converse relation $\breve{R}$; the converse of the converse of a relation is equivvalent to the relation, that is, $\breve{\breve{R}} = R$; if $R_1 = \breve{R}_2$, then $\breve{\rho}_1 = \rho_2$, and $\rho_1 = \breve{\rho}_2$, and, if the latter two equivalences subsist, then $R_1 = R_2$; also, if $R_1 = \breve{R}_2$, then $\breve{R}_1 = R_2$; the converse of the relative product of two relations is equivalent to the relative product of their converses reversed in order, that is $(\breve{R_1R_2}) = \breve{R}_2\breve{R}_1$; if R is transitive and if xRz, there exists a y such that xRy and yRz; the converse of the negative of a relation is equivalent to the negative of the converse of the relation; a null-class is included in every other class; if, for every x in the domain ρ of

R, xRy is equivalent to $y\epsilon x$, then $R = \breve{\epsilon}$; if u and v are existent (not null) classes, there exists a relation subsisting between every term of u and every term of v but not between other two terms; if u is an existent class, there exists a relation R such that xRu implies for every x both $\rho = u$ and $x\epsilon u$, and, conversely, the product of $\rho = u$ and $x\epsilon u$ implies xRu for every x; identity is transitive; identity is equivalent to its converse; the relative product of identity by itself is equivalent to identity; diversity is equivalent to the converse of diversity; if $R_1\breve{R}_2$ is included in diversity, so is $\breve{R}_1R_2$, and conversely; identity is biuniform; if a relation is biuniform, so is its converse; if a relation is couniform, the relative product of it and its converse is included in but is not always identical with identity; if two relations are biuniform, so is their relative product; given that R_1 and R_2 are uniform relations, that u is a class included in ρ_1, that $\breve{\rho}u$ is included in ρ_2 and that $R_1R_2 = R$, then the two classes, $\breve{\rho}_2(\rho_1 u)$ and $\breve{\rho}u$, are equivalent; if R_1 is uniform and if $R_2 = R_1\breve{R}_1$, then R_2 is transitive and symmetric; conversely, *if an existent relation R_2 is transitive and symmetric, then there exists a uniform relation R_1 such that $R_2 = R_1\breve{R}_1$.*

So striking as well as important is the theorem last stated that I can not refrain from presenting its demonstration, which runs as follows: R_2 being given, ρ_2 is also given: let x be a term of ρ_2, and denote by u the class $\breve{\rho}_2 x$; let R_1 be such that xR_1u means $x\epsilon\breve{\rho}_2$ and $u = \breve{\rho}_2 x$; then, if yR_1u, $y\epsilon\rho_2$ and $u = \breve{\rho}_2 y = \breve{\rho}_2 x$; but, if xR_1u and yR_1u, then, $xR_1\breve{R}_2y$; and, as R_2 is transitive and symmetric, xR_2y; hence, as $xR_1\breve{R}_1y$ implies xR_2y, $R_1\breve{R}_1$ is included in R_2; again, as R_2 is transitive and symmetric, if xR_2y then $x\epsilon\breve{\rho}x$, and so xR_2y implies $xR_1\breve{\rho}x$ and $yR_1\breve{\rho}x$, and hence implies $xR_1\breve{R}_1y$; hence R_2 is included in $R_1\breve{R}_1$; hence $R_2 = R_1\breve{R}_1$; moreover, R_1 is uniform, its codomain consisting of the single term u. Hence the theorem.

As in the case of propositions and in that of classes, so here, too, are valid the theorems of tautology, association, commutation, distribution and double negation:

$$R \frown R = R = R \smile R;$$

$$(R_1 \frown R_2) \frown R_3 = R_1 \frown (R_2 \frown R_3),$$

$$(R_1 \smile R_2) \smile R_3 = R_1 \smile (R_2 \smile R_3);$$

$$R_1 \frown R_2 = R_2 \frown R_1,\ R_1 \smile R_2 = R_2 \smile R_1;$$

$$R_1 \frown (R_2 \smile R_3) = (R_1 \frown R_2) \smile (R_1 \frown R_3),$$

$$R_1 \smile (R_2 \smile R_3) = (R_1 \smile R_2) \smile (R_1 \smile R_3);$$

$$-(-R) = R.$$

Awhile ago I promised to "explicate" the thesis of modern logistic, to state it, that is, explicitly in terms of the logical primitives upon which as the sufficient foundation it asserts that the entire body of mathematics, both actual and potential, stands as a superstructure. The primitives in question have been given; so that, except for a restatement of the thesis in terms of them—which I shall omit as being now easy and involving useless repetition—I may claim to have done much more than fulfil the promise; for I have given in addition to the primitives, which were all that was essential, a digest of modern logic. Indeed, the concepts above defined and the theorems above stated, though they are conventionally assigned to logic, are evidently, if the thesis be true, genuine parts of mathematics.

How is the thesis, if true, to be established? Obviously not, in the ordinary sense, as the conclusion of a syllogism. No, it affirms that a certain thing can be done, namely, that all definable mathematical ideas and all mathematical theorems are respectively definable and demonstrable in terms of the primitives given. The only way to show that the deed is

performable is to perfrom it. Here nothing can succeed except success. Happily the procedure in question need not be applied to *all* mathematical concepts and theorems but only to those—and they are not so numerous—upon which, it is admitted, the remainder rest. Well, an examination of the volumes of the *Revista di Matematica* and of its continuation, the *Revue de Mathematiques,* will show that the principal mathematical branches have been successfully subjected to the treatment in question, with reference, however, to primitive-systems differing somewhat from that above given. As for the latter system, its adequacy to the demands of the thesis has been shown by Russell in his "Principles" with approximate completeness and with as much rigor as discourse, mainly non-symbolic, can be reasonably expected to attain. If, as is to be expected, new branches of mathematics shall arise in the days to come, though we can not be absolutely certain, we may confidently expect that they will be congruous with existing doctrines and will not demand a radical change in foundations.

Process of Testing the Thesis Illustrated.—The little time that remains to me for this address, I shall devote to illustrating by means of a few cardinal examples, the procedure by which the thesis is justified. And I shall begin with the concept of *cardinal number.* Before defining *cardinal number of a class,* we define what is meant by *sameness* of cardinal number, or, better, what is meant by saying this class and that have the same cardinal number. Two classes a and b are said to have the same cardinal number when there is a biuniform relation, or, as we commonly phrase it, a one-one correlation between them. A slight change in the statement is necessary to prove suitable for zero. Then the cardinal number of a class a is defined to be *the class* whose terms are the classes having each of them, according to the preceding definition, the same cardinal number as a. Thus with each class is associated a definite cardinal number. That of the null-class is named *zero* and denoted by 0; that of a singular class is called *one* and denoted by 1. Addition of cardinals is definable in terms of logical addition of classes: if a and b be two disjoint classes having respectively the numbers α and β, the sum $\alpha + \beta$ is the number of the logical sum (a class) $a + b$ of a and b. If a and b are singular classes, the cardinal of their sum may be named *two* and denoted by the symbol *2*, in which case $1 + 1 = 2$; and so on. *Multiplication* of cardinals is also defined in purely logical terms. This is done by means of the concept (due to Whitehead) of *multiplicative* class, which is itself given in terms of logical constants: k being a class of disjoint classes, the *multiplicative class* of k is the class of all the classes each of which contains one and but one term of each class in k. Then the *product* of the cardinal numbers of the classes in k is defined to be the cardinal number of the multiplicative class of k. As multiplication and addition in class logic are commutative, associative and distributive, it readily follows that these laws are valid for cardinal numbers. In the manner indicated the entire theory of cardinals can be established. And thus it appears—to refer again to an example before cited—that the foundation assumed by Weierstrass for the theory of the real variable is itself underlaid by a basis in pure logic.

It is noteworthy that the foregoing concept of cardinal is independent of the (as yet undefined) notion called *order* and that it equally comprises both *finite* and *infinite* cardinals, the distinction of finite and infinite being this: the cardinal number of a class a is infinite or finite according as a is or is not such that there is a class b com-

posed of some but not all of the terms of a and having to a a biuniform relation. In respect to the finite cardinals, they may be defined as follows, presenting them in what, once order is defined, will be called a *series,* 0, 1, 2, . . . Let zero (0) be defined as above; let the *cardinal next after the cardinal* n be defined to be the cardinal $n+1$; let N, the class of finite cardinals, be defined to be the class of cardinals that are contained in every class that contains 0 and contains $n+1$ if it contains n. It remains then to show that the two definitions of finite cardinals are equivalent, and that can be done.

Cardinals, we have seen, are *classes.* The ordinary rational numbers, or fractions, are not classes, but are, as we shall see, *relations* of finite cardinals. Let a be any given finite cardinal, and let x and y be any finite cardinals such that $xa=y$. Denote by A the relation such that xAy is equivalent to $xa=y$. Similarly, to any finite cardinal n there corresponds a relation N whose domain and codomain are respectively composed of all the finite cardinals x and y such that $xn=y$. If $ab=p$ and $cd=p$, that is, if $ab=cd$, then aBp and cDp, whence $p\breve{D}c$, so that $aB\breve{D}c$. The relation $B\breve{D}$, the relative product of B and the converse of D, is named rational number, or fraction, and denoted by b/d. If $ab=cd$, it readily follows that $b/d=a/c$. The rational $n/1$ is commonly denoted by n, but the rational n and the cardinal n are radically different, the former being a relation while the latter is a class.

The cardinals and rationals are signless. Like the rationals, positive and negative integers and fractions are relations but they are relations of a different type. Suppose the finite cardinals arranged as by their second definition above given. Let R be such that xRy, x and y being finite cardinals, means that, in the mentioned arrangement, y is the immediate successor of x; then $x\breve{R}y$ means that y is the immediate predecessor of x. It is readily proved that R^p is the converse of $(\breve{R})^p$ or, what is the same, of $\breve{R}^p$. The relations R^p and $\breve{R}^p$ (p being a finite cardinal) are defined to be the positive and negative integers familiarly denoted by $+p$ and $-p$ respectively. Thus to each finite cardinal p there corresponds a positive integer, $+p$, and a negative integer, $-p$. If x, y and p are finite cardinals, the propositions, xR^py and $x+p=y$, are equivalent; so, too, are $x\breve{R}^py$ and $y+p=x$ or $x-p=y$. Similarly if x be a rational number, and if y and z stand for any two rational numbers so related that $y+x=z$, the relation in question is denoted by $+x$; but if y and z are so related that $y-x=z$, the relation is denoted by $-x$.

Before speaking of the *ordinal number,* it is necessary to tell what is meant by saying of a class that it is ordered or that its terms are arranged in a *series.* This, which is one of Russell's most brilliant achievements, was accomplished as follows. I here but indicate the method and state the result. The method was precisely that of research in natural science, namely, he collected together the various kinds of relation by which what is called order, whatever order in its essence should turn out to be, is generated. These relations, which he found to belong to one or another of six distinct types, turned out, upon penetrating analysis, to be reducible to a single type, namely, that of relations at once *transitive* and *asymmetric,* an asymmetric relation R being such that, if xRy, then not yRx. The conclusion may be stated to be that, a class being given, if there exist a transitive asymmetric relation R such that, x and y being any two whatever of its terms, either xRy or else yRx, the class is

thus arranged in a *series;* and that order otherwise generable is generable by such a relation. The result is of course subject to such doubt as must always attend the method employed, but its correctness seems highly probable. It can be easily proved that, given any three terms x, y, z of an open series, we have xRy and yRz, or yRz and zRx or zRx and xRy, that is, one of the three terms is *between* the other two; and if the series be closed, like that of the points of a circle, it can be rendered open by *cutting* it—that is, by regarding it as beginning (or ending) with some (any) definite term.

We are now prepared to present the notion of ordinal number. If, given two series s_1 and s_2, there subsist between them, regarded as classes, a biuniform relation R such that, a_1 and b_1 being any two terms of s_1 and a_2 and b_2 their respective correspondents (through R) in S_2, a_1 precedes or follows b_1 according as a_2 precedes or follows b_2, then the series s_1 and s_2 are said to be *like.* Plainly likeness is a transitive and symmetric relation. Two like series are said to have the *same ordinal number* or the *same order-type.* Herewith ordinal number, or order-type, of a series is yet not defined. The definition is: the ordinal number, or order-type, of a series s is the *class* of all series like it. Or, defining *like* relations to be such as generate like series, we can define ordinal number, or order-type, of a series-generating relation to be the class (a relation by primitive proposition) of series like it. The definition does not distinguish finite and infinite and so applies to both. In case the terms of a series constitute a finite class, the cardinal number of the class and the ordinal number of the series obey the same laws and are commonly denoted by the same name and symbol. Yet they are radically different notions. For example, the *cardinal three* includes the class composed of a, b and c, but not the series a, b and c as such, while the *ordinal three* includes the series but not the class. On transition to infinites the distinction is forced upon us, for infinite cardinals obey, for example, the law of commutation, while the infinite ordinals do not.

I have time for but a single indication pointing the way to the concept and theory of *real* numbers. Consider, for example, the two familiar classes: A, the class of rationals less than 2; B, the class of rationals whose squares are less than 2. Each of these classes possesses the properties: (1) it does not contain all the rational numbers; (2) it contains all the rational numbers less than any one of its numbers; (3) every number in it is less than some other number in it. Any class of rationals that has the three properties is named *segment* (of rationals). Given a segment s, the class of rationals not belonging to s may be called the *cosegment* of s. It is found that the class of all segments admits of a theory precisely isomorphic with that of the real numbers as usually defined. Hence the segments are named *real numbers.* Segments fall into two classes according as their cosegments have or have not a smallest rational. In the former case the segment is called a *rational* real number. Thus segment A is the rational real *two* or 2. In the other case, the segment is called an *irrational* real number. Thus segment B is the irrational real commonly denoted by $\sqrt{2}$. It is obvious that segments and reals might just as well be defined by the relation greater than instead of less than. The decisive advantage of the foregoing definition, which makes no appeal to the (as yet) undefined notion of *limit,* is that it avoids the necessity of *assuming* a limit where there is none, as in case of class B.

It is to be noted that in usage various kinds of numbers are denoted by the same symbol. This is due to the fact that custom antedates criticism. Thus 2 stands for a cardinal (a class), for a positive integer (a relation), for a rational number or fraction (a relation), for an ordinal (a relation), and for a rational real (a class)—neither the classes nor the relations being of the same kind.

Passing now to the notion of the (linear) *continuum*, it is to be defined in ordinal terms and without the logically vicious assumption often tacitly made that the continuum to be defined is already immersed in a continuum. The following procedure is due to G. Cantor. Let η denote the order-type of series *like* that of the rationals taken in so-called natural order. Any series of this type has the following properties, all of them ordinal: (1) it is denumerable; (2) it has neither beginning nor end; (3) it is compact. A series of terms in a series of type η is said to be *fundamental* if it is a *progression*, that is, if it is like the series 1, 2, 3, $\cdots$; and it is described as *ascending* or *descending* according as its terms follow one another in the same sense (or direction) as do those of the series η or in the reverse sense. A term of a series is a *limit* if it immediately follows (or precedes) a class of terms of the series and does not immediately follow (or precede) any one assignable term of it. It follows that a fundamental series s of a series η has a limit if in η there is a term that is first after or first before all the terms of s according as s is ascending or descending. A series is said to be *perfect* if (1) all its fundamental series have limits and (2) all its terms are limits of fundamental series. It can be proved that a series whose terms are terms of a perfect series and which, besides being denumerable, are so distributed that there is one between every two terms of the perfect series, is a series of type η. We can now define: a series θ is *continuous* if it is perfect and contains a denumerable class of terms such that there is one of them between every two terms of θ. The definition is based upon the properties found to characterize the series of real numbers from 0 inclusive to 1 inclusive.

The significance of what has been said is by no means confined to analysis. Yet I wish, in closing, to refer explicitly to geometry. As a branch of mathematics, geometry does not claim to be an accurate or true description of actual or perceptual space, whatever that may be. As for the notion and the name of space, it does not seem to be a *modern* discovery that they are not essential to geometry, for, as Peano has pointed out, neither the one nor the other is to be found in the works either of Euclid or of Archimedes. What, then, is geometry? And how related to the thesis of modern logistic? The answer must be in terms of *form* and *subject-matter*. As to form, geometry is, as Pieri has said and by his great memoirs has done as much as any one to show, a purely "hypothetico-deductive" science. It is true indeed that in each of the postulate-systems—whether those of Pieri or of Pasch or of Peano or of Hilbert or of Veblen or of others—that have recently been offered as basis for descriptive or projective or metric geometry or for any sub-division of those grand divisions, there occurs at least one postulate in categoric form, as, for example, "there exists at least one point"—thus seeming to assert or to imply that the geometry in question, whatever variety it may be, transcends the hypothetic character and has in fact validity of an extra-theoretic or external kind. Nevertheless, the seeming is appearance only. What the geometrician really asserts, and he asserts nothing

else, is that, *if* there be terms, which he calls points, and might as well call "roints" or "raths" or "momes" or any other name (what's in a name?), that satisfy the given postulates, then they satisfy certain propositions called theorems. The only existence asserted by or in geometry is thus the existence of certain *implications*. As to subject-matter, that of geometry, as Russell has, I think, shown beyond a reasonable doubt, is multiple series or, more radically, the relations by which such series are generated or in which they extensionally consist.

I wish to add in closing that this address had not been possible but for the far-reaching researches and brilliant expositions of Schröder, Russell and Couturat in the works already cited.

C. J. KEYSER

COLUMBIA UNIVERSITY

CHEMISTRY AT HARVARD UNIVERSITY

THE following letter has been prepared by the committee of overseers to visit the chemical laboratory of Harvard University and by several others who are especially interested in the subject:

HARVARD UNIVERSITY is in urgent need of the endowment of modern facilities for chemical instruction and research.

Some progress toward such an endowment has already been made by the conditional offer of contributions for the construction of a special laboratory for research in physical and inorganic chemistry, as a memorial to Wolcott Gibbs.

Wolcott Gibbs was a pioneer in scientific research in the field of inorganic and physical chemistry, and for many years was considered the foremost chemist of America. He died on December 9, 1908, in his eighty-seventh year. The greater part of his useful life was spent as Rumford professor at Harvard University, and it is eminently fitting that any memorial to this great and good man should take a form which would further that branch of chemistry to which he had devoted his splendid abilities.

This project forms a highly suitable beginning of the much-needed endowment of modern facilities for chemical instruction and research at Harvard University, because in precise investigations of this kind Harvard is among the leading institutions of the world. Such work demands, for its highest development, construction and facilities superior to any now in existence; and above all this laboratory should be designed for research only, and separated from the rooms in which elementary teaching is conducted. The new building would also partially relieve the very disadvantageous and unhygienic condition of Boylston Hall, now one of the most crying evils in Harvard University.

This Wolcott Gibbs Memorial Laboratory would form part of the group of several buildings necessary for the adequate accommodation of the department of chemistry. The report of the Committee of Overseers to Visit the Chemical Laboratory contains a provisional plan of this projected group, which offers a magnificent opportunity for other large gifts. These would form dignified memorials of benefactors or those named by them, as well as permanent sources of usefulness to Harvard and to America.

The report just mentioned calls attention to the important rôle played by pure chemistry in almost all departments of industrial science which contribute towards the health and prosperity of mankind, and concludes:

"The last century has been a century of power, by the perfection of machinery and the development of electricity. The coming century promises to be a chemical century. Should Harvard, if all this be true, be content until it has obtained the best chemical laboratory in the world?"

Towards the erection of the Wolcott Gibbs Memorial Laboratory subscriptions of nearly $53,000 have already been made, most of them upon the condition that $47,000 more be immediately secured. Checks either for this fund or as contributions toward one of the other laboratory buildings may be drawn to the order of Charles Francis Adams, 2d, treasurer of Harvard College, 50 State Street, Boston.

J. COLLINS WARREN,	CHARLES W. ELIOT,
JAMES M. CRAFTS,	ALEXANDER AGASSIZ,
ELIHU THOMSON,	HENRY P. WALCOTT,
E. D. PEARCE,	HENRY L. HIGGINSON,
CLIFFORD RICHARDSON,	ALEXANDER COCHRANE,
CHARLES H. W. FOSTER,	FREDERICK P. FISH,
MORRIS LOEB,	HARRISON S. MORRIS,
A. LAWRENCE LOWELL,	E. MALLINCKRODT, JR.,

Committee of the Overseers to Visit the Chemical Laboratory

President Lowell's interest is emphatically expressed in the following letter, which he kindly permits to be published: